

Fractal dimensions of strange attractors obtained from the Taylor–Couette experiment

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We present scenarios from the rotational Taylor–Couette flow, which has been implemented as a high precision hydrodynamic experiment. The system shows a rich variety of routes to chaos, e.g. the period doubling cascade, quasiperiodic and intermittent transitions. From a scalar time series, that is the axial velocity component of the flow field measured with Laser–Doppler velocimetry, strange attractors are reconstructed using Takens' delay time coordinates. To obtain an estimate of the fractal dimensions of these structures in phase space we calculate the correlation dimension from the reconstructed attractors. We discuss the fractal dimension as a function of Reynolds number and geometry of the experiment.

1. Introduction

The fractal structure of turbulence was first mentioned by Mandelbrot [1]. While turbulence is a spatial-temporal effect, in this paper we focus on chaotic states of a Taylor–Couette flow. Calculations of fractal dimensions to analyze the properties of this dissipative nonlinear dynamical system have become a well-established method. The dimensions are calculated from attractive sets with very complicated structures in phase space. The so-called strange attractors show sensitive dependence on the initial conditions, i.e. initially adjacent points separate exponentially for sufficiently large time. Due to dissipation, volumes in phase space shrink and therefore lengths cannot expand in all directions. Because of a finite boundary of the attractive basin, volume elements are folded at the same time. This stretching, shrinking and folding process leads to a self-similar structure of the attractor.

Unfortunately, in most experimental situations only one observable is available (in the Taylor–Couette experiment, e.g., the axial velocity component $v_z(t)$), so the phase space must be reconstructed from the scalar time series. Takens's delay time coordinates method [2] is the one commonly used. A vector in the reconstructed space is given by $x(t) = (v_z(t), v_z(t + \tau), \dots, v_z(t + \tau^*(\dim_E - 1)))$, where the embedding parameters, delay time τ and embedding dimension \dim_E , must be chosen carefully [3].

To estimate the fractal dimension of the reconstructed strange attractors in phase space we calculate the correlation dimension D_2 [4],

$$C(R) \propto R^{D_2} \Rightarrow D_2 = \lim_{R \rightarrow 0} \frac{\log_{10}[C(R)]}{\log_{10}(R)}. \quad (1)$$

R is the scaling radius and $C(R)$ is the correlation integral

$$C(R) \approx \frac{1}{N_{\text{ref}}} \sum_{j=1}^{N_{\text{ref}}} \frac{1}{N_{\text{dat}}} \sum_{i=1}^{N_{\text{dat}}} \sigma(R - \|x_i - x_j\|), \quad (2)$$

where σ is the Heaviside function, N_{dat} is the number of points in phase space and N_{ref} a sufficiently large number of reference points.

The attractors presented below were obtained from a Taylor–Couette system, a flow of a viscous fluid between two coaxial cylinders. The inner cylinder ($r_i = 12.5$ mm) is rotating; the outer cylinder ($r_o = 25.0$ mm) as well as bottom and top are at rest. The gap length h can be adjusted continuously from $\Gamma = h/d = 0$ to $\Gamma = 48$ ($d = r_o - r_i$). The control parameter is the Reynolds number $\text{Re} = \Omega r_i d / \nu$, where Ω is the rotation frequency and ν the kinematic viscosity.

2. Experimental results

(a) Period doubling cascade

For very small geometries ($0.3 \leq \Gamma \leq 0.6$) one finds a symmetric two-vortex flow showing a Hopf bifurcation. In fig. 1 the extrema of this period-one mode P_1 are plotted for $\Gamma = 0.374$. At $\text{Re} = 663$ a period doubling bifurcation appears, leading to a period-two mode P_2 . At $\text{Re}_c \approx 691$ the transition to chaos occurs. The envelope of the bifurcation diagram grows monotonously when the

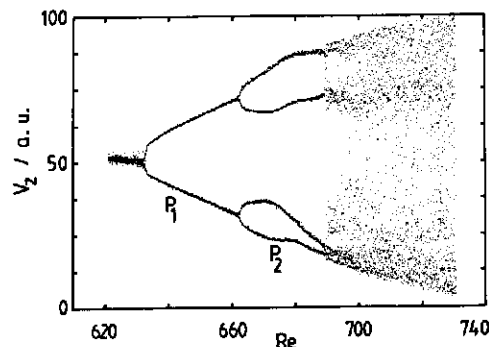


Fig. 1. Bifurcation diagram obtained from plotting consecutive extrema of the time series, while the Reynolds number is continuously ramped from $\text{Re} = 620$ to 730 .

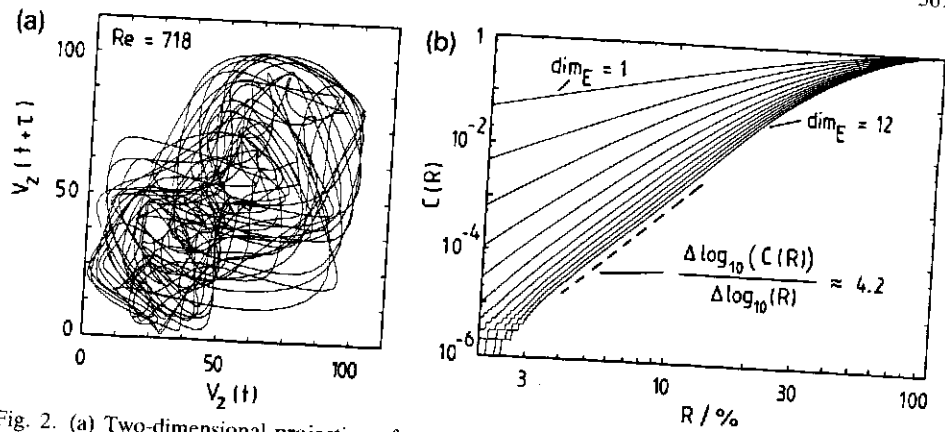


Fig. 2. (a) Two-dimensional projection of a reconstructed strange attractor for Reynolds number $Re = 718$. (b) Double-logarithmic plot of the correlation integral versus scaling radius R from embedding dimension $dim_E = 1$ to 12. The slope of the dashed line shows the estimated fractal dimension.

Reynolds number is increased up to $Re \approx 730$. At this point the symmetric two-vortex flow becomes unstable and an oscillatory asymmetric one-vortex flow is established [5], which will not be discussed here. Fig. 2a shows a reconstructed strange attractor representing the chaotic state for $Re \approx 718$. The self-similar structure is hidden in this plot, because it is a projection from a high-dimensional embedding and the trajectories are covered with noise. Fig. 2b shows the double-logarithmic plot of the correlation integral $C(R)$ versus radius R for embedding dimensions up to $dim_E = 12$. For $dim_E > 7$ the slopes of the curves are constant for the relevant scaling interval yielding a correlation dimension $D_2 \approx 4.2$.

Calculating the dimensions for the measured control parameter range Re , we found a transition to chaos following $D_2 \propto (Re - Re_c)^{1/4}$, very similar to a continuous phase transition. In fig. 3 the estimates of the correlation dimension and the corresponding error bars are plotted versus Reynolds number.

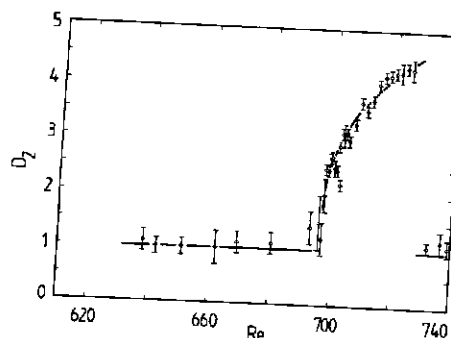


Fig. 3. Estimates of fractal dimensions of the strange attractors versus Reynolds number.

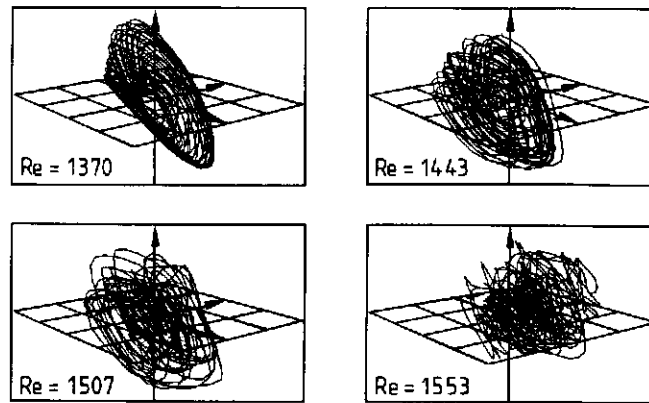


Fig. 4. Projection of reconstructed attractors from the quasiperiodic scenario showing a break-up of a two-torus for Re : 1370, 1443, 1507 and 1553.

(b) *Quasiperiodic transition*

For larger geometries ($\Gamma = 7.47$) one finds a two-mode state in an eight-vortex flow that depends on the stabilization of a local Große-Jet mode, which is a local oscillation of the outward flow between two vortices with a very large amplitude, and an Anti-Jet mode, an oscillation of the inward flow. This two-mode state shows a transition to chaos at $Re_c \approx 1400$ illustrated in fig. 4 by four reconstructed attractors. In fig. 5 the linear increase of the fractal dimension is given by $D_2 \propto Re - Re_c$. Due to noise the calculated values of the correlation dimension for the laminar regime are larger than expected (indicated by the dashed line in fig. 5). This break-up of a two-torus is very similar to the results of Brandstätter and Swinney [6], though we must state that there is no unique Taylor–Couette attractor but many scenarios have been found [7].

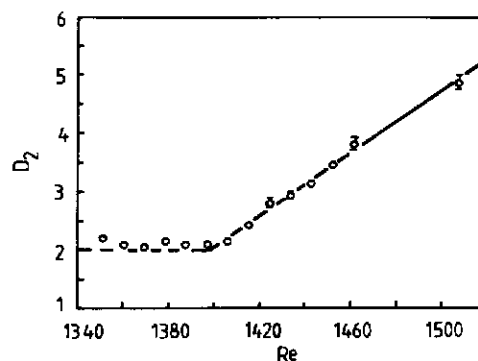


Fig. 5. Estimates of fractal dimensions of the strange attractors versus Reynolds number for the break-up of a two-torus.

3. Conclusion

Two scenarios obtained from the Taylor-Couette experiment are presented. We show that the evolution of the fractal dimension of the considered routes depends on the system's boundary conditions. One problem in finding an accurate fractal dimension is that trajectories of experimental attractors are always covered with noise. We are working on noise reduction procedures in phase space to find the underlying "correct" fractal dimension.

Our main concern is the transition from order to disorder, i.e. in our case the change from laminar to chaotic flow. The fractal dimension is used as an order parameter which characterizes these transitions.

References

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