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The geometry of second symmetric products of curves

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This paper is a short summary of the main results in the thesis [1] and in the paper [40]. We deal throughout with several problems on the surfaces obtained as second symmetric products of smooth projective curves. In particular, we treat both some attempts at extending the notion of gonality for curves and some classical problems, as the study of the ample cone in the Néron-Severi group. Moreover, we develop a family of examples of Lagrangian surfaces having particular topological properties.

1 Introduction

Let C be a smooth irreducible complex projective curve of genus $g \geq 0$. We define the *gonality of* C as the minimum integer d such that the curve admits a covering $f\colon C \longrightarrow \mathbb{P}^1$ of degree d on the projective line and we denote it by $gon(C)$. The problem of estimating the gonality of a smooth curve is a classical issue in the theory of algebraic curves which has been studied - for instance - by Brill-Noether theory (cf. [3, Chapter IV]). Let us denote by $C^{(2)}$ the second symmetric product of C, which is the smooth surface parametrizing the unordered pairs of points of C. Our first aim is to present two attempts to extend the notion of gonality to varieties of higher dimension and in the next sections we proceed to state the main results on this topic in the case of second symmetric products of curves.

A first generalization of gonality is the notion of degree of irrationality introduced by Moh and Heinzer in [4], which has been deeply studied by Yoshihara (see [5–7]). If X is an irreducible complex projective variety of dimension n, the *degree of irrationality of* X is defined to be the integer

$$d_r(X) := min\left\{ d \in \mathbb{N} \;\middle|\; \begin{array}{l} \text{there exists a dominant rational map} \\ F\colon X \dashrightarrow \mathbb{P}^n \text{ of degree } d \end{array} \right\}.$$

This number is clearly a birational invariant, and having $d_r(X) = 1$ is equivalent to rationality. Moreover, as any dominant rational map $f\colon C \dashrightarrow \mathbb{P}^1$ can be resolved to a morphism, it follows that $d_r(C) = gon(C)$ and hence the degree of irrationality does provide an extension of gonality to n-dimensional varieties.

We would like to recall that if there exists a dominant rational map $C \dashrightarrow C'$ between curves, then $gon(C) \geq gon(C')$. On the other hand, the existence of a dominant rational map $X \dashrightarrow Y$ between varieties of dimension $n \geq 2$, does not lead to an analogous inequality for the degrees of irrationality. Indeed there are counterexamples in the case of surfaces (cf. [6, 17]) and there are examples of non-rational threefolds that are unirational (see for instance [9, 10]).

Another proposal at extending the notion of gonality to n-dimensional varieties is the following. Given an irreducible complex projective variety X, we define the number

$$d_o(X) := min\left\{ d \in \mathbb{N} \;\middle|\; \begin{array}{l} \text{there exists a family } \mathcal{E} = \{E_t\}_{t\in T} \\ \text{covering } X \text{ whose generic member is} \\ \text{an irreducible } d\text{-gonal curve} \end{array} \right\}$$

and we may call it the *degree of gonality of* X. Notice that the generic member E_t is a possibly singular curve, hence we refer to the gonality of its normalization $\widetilde{E}_t \longrightarrow E_t$. The degree of gonality is a birational invariant, and $d_o(X) = 1$ if and only if X is an uniruled variety. Moreover, $d_o(C) = gon(C)$ for any complex projective curve C.

Although this second generalization of the notion of gonality appears less intuitive and more artificial than the degree of irrationality, the degree of gonality has a nice behavior with respect to dominance. Namely, if there exists a dominant rational map $X \dashrightarrow Y$ between two irreducible complex projective varieties of dimension n, it is easy to see that $d_o(X) \geq d_o(Y)$, as in the one-dimensional case.

We note that another attempt to extend the notion of gonality has been recently presented in terms of pencils and fibrations (cf. [11]).

Section 2 of this paper concerns the study of the degree of irrationality of second symmetric products of curves. Moreover, we shall briefly present the techniques involved in the proofs of the main results.

In Section 3 we shall focus on deformation and gonality of curves lying on second symmetric products. On one hand, we shall discuss the existence of hyperelliptic curves lying on the Jacobian variety - and a fortiori on the second symmetric product - of a generic curve by extending a result of Pirola (see [12]). On the other hand, we shall treat the problem of determining the degree of gonality of second symmetric products of curves. As an application of the latter study, by improving a result of Ein and Lazarsfeld (see [13, Corollary 1.2]) and by following the argument of Ross in [54, Section 4], we shall give new bounds on the ample cone of second symmetric product of a generic curve of genus $5 \leq g \leq 8$.

Finally, in Section 4 we shall turn to Lagrangian surfaces. In particular, we shall sketch out the construction of a family of examples of Lagrangian surfaces with negative topological index. We would like to note that we computed the other birational invariants as well and that this construction applies also to the second symmetric product of particular curves of genus 2.

Notation. We work throughout over the field $\mathbb{C}$ of complex numbers. Given a variety X, we say that a property holds for a *general* point $x \in X$ if it holds on an open non-empty subset of X. Moreover, we say that $x \in X$ is a *very general* point if there exists a countable collection of proper subvarieties of X such that x is not contained in the union of those subvarieties. By *curve* we mean a complete reduced algebraic one-dimensional variety over the field of complex numbers. When we speak of *smooth curve*, we always implicitly assume it to be irreducible.

2 Degree of irrationality of second symmetric products of curves

Let C be a smooth complex projective curve of genus $g \geq 0$ and let $C^{(2)}$ be its second symmetric product. As we anticipated, our aim is now to estimate the degree of irrationality $d_r\left(C^{(2)}\right)$ of the latter surface.

We would like to note that there is a strong connection between the existence of a dominant rational map $F\colon C^{(2)} \dashrightarrow \mathbb{P}^2$ and the genus g of the curve C. For instance, it is immediate to check that rational and elliptic curves are such that the degree of irrationality of their second symmetric product is one and two respectively, whereas we proved that if the genus of C is $g \geq 2$, then $C^{(2)}$ is non-rational and it does not admit a dominant rational map on $\mathbb{P}^2$ of degree 2, that is $d_r\left(C^{(2)}\right) \geq 3$. In particular, when $g \geq 2$ the problem of computing $d_r\left(C^{(2)}\right)$ is still open.

Furthermore, the degree of irrationality of the second symmetric product seems to depend on the existence of linear series on the curve as well. In particular, if C admits a degree d covering $f\colon C \longrightarrow \mathbb{P}^1$, it is always possible to define a morphism $C^{(2)} \longrightarrow \mathbb{P}^2$ of degree d^2 by sending a point $p+q \in C^{(2)}$ to the point $f(p)+f(q) \in (\mathbb{P}^1)^{(2)} \cong \mathbb{P}^2$. Hence the degree of irrationality of the second symmetric product of a curve is bounded from above by the square of the gonality. Moreover, if C admits a birational mapping onto a non-degenerate curve of degree d in $\mathbb{P}^2$ we may construct a dominant rational map $C^{(2)} \dashrightarrow \mathbb{P}^2$ of degree $\binom{d}{2}$, which sends a point $p+q \in C^{(2)}$ to the line $l \in \mathbb{G}(1,2) \cong \mathbb{P}^2$ passing through the images of p and q in $\mathbb{P}^2$. Finally, it is possible to provide other dominant rational maps by using birational mapping onto a non-degenerate curves of $\mathbb{P}^3$ as well. Thus we have the following upper bound.

Proposition 2.1 *Let C be a smooth complex projective curve. Let δ_1 be the gonality of C and for $m = 2, 3$, let δ_m be the minimum of the integers d such that C admits a birational mapping onto a* *non-degenerate curve of degree* d *in* $\mathbb{P}^m$*. Then*

$$d_r(C^{(2)}) \leq min \left\{ \delta_1^2 \, , \, \frac{\delta_2(\delta_2 - 1)}{2} \, , \, \frac{(\delta_3 - 1)(\delta_3 - 2)}{2} - g \right\} .$$

In the case of hyperelliptic curves we prove the following.

Theorem 2.2 *Let* C *be a smooth complex projective curve of genus* $g \geq 2$ *and assume that* C *is hyperelliptic. Then*

(i) $3 \leq d_r(C^{(2)}) \leq 4$ *when either* $g = 2$ *or* $g = 3$*;*

(ii) $d_r(C^{(2)}) = 4$ *for any* $g \geq 4$*.*

We point out that there are examples of curves of genus two with degree of irrationality equal to 3 (cf. [7, Theorem 0.2] and Section 4), but these constructions do not apply to generic curves. On the other hand, the theorem above assures that if when $g \geq 4$, the degree of irrationality is as great as possible.

When the curve is assumed to be non-hyperelliptic, the situation is more subtle and it is no longer true that the degree of irrationality of $C^{(2)}$ equals the square of the gonality of C for high enough genus. The following result summarizes the lower bounds we prove on the degree of irrationality of second symmetric products of non-hyperelliptic curves and we list them by genus.

Theorem 2.3 *Let* C *be a smooth complex projective curve of genus* $g \geq 3$ *and assume that* C *is non-hyperelliptic. Then the following hold:*

(i) *if* $g = 3, 4$*, then* $d_r(C^{(2)}) \geq 3$*;*

(ii) *if* $g = 5$*, then* $d_r(C^{(2)}) \geq 4$*;*

(iii) *if* $g = 6$*, then* $d_r(C^{(2)}) \geq 5$*;*

(iv) *if* $g \geq 7$*, then*

$$d_r(C^{(2)}) \geq max \{ 6, \, gon(C) \} .$$

Furthermore, if C *is assumed to be very general in the moduli space* $\mathcal{M}_g$ *with* $g \geq 4$*, then*

$$d_r(C^{(2)}) \geq g - 1.$$

As regard of the latter inequality, we note that when C is a very general curve, the minimum degrees we are able to present for dominant rational maps $C^{(2)} \dashrightarrow \mathbb{P}^2$ are those listed in Proposition 2.1. Furthermore - except for finitely many genera - the lowest of them is the bound depending on δ_2. Finally, we note that analogous remarks hold for k-fold symmetric products of curves $C^{(k)}$ with respect to δ_k, with $k \geq 3$. Therefore, it seems natural to conjecture that given a very general curve C of high enough genus, the the degree of irrationality of $C^{(2)}$ is $\frac{\delta_2(\delta_2-1)}{2}$, but Theorem 2.3 is at the moment our best bound.

In order to prove the most of these results, the main technique is to use holomorphic differentials, following Mumford's method of induced differentials (cf. [15, Section 2]). In the spirit of [16], we rephrase our settings in terms of correspondences on the product $Y \times C^{(2)}$, where Y is an appropriate ruled surface. A general 0-cycle of such a correspondence $\Gamma \subset Y \times C^{(2)}$ is a Cayley-Bacharach scheme with respect to the canonical linear series $|K_{C^{(2)}}|$, that is any holomorphic 2-form vanishing on all but one the points of a 0-cycle vanishes in the remaining point as well. The latter property imposes strong conditions on the correspondence Γ, and the crucial point is to study the restrictions descending to the second symmetric product and then to the curve C. This approach leaded us to deal with sets of lines of $\mathbb{P}^n$ satisfying a condition of Cayley-Bacharach type. Namely, we proved the following.

Theorem 2.4 *Let $l_1, \dots, l_d \subset \mathbb{P}^n$ be lines enjoying the property that for every $i = 1, \dots, d$ and for any $(n-2)$-plane $L \subset \mathbb{P}^n$ intersecting $l_1, \dots, \widehat{l_i}, \dots, l_d$, we have $l_i \cap L \neq \emptyset$ too. Then the dimension of their linear span $S = Span(l_1, \dots, l_d)$ in $\mathbb{P}^n$ is $s \leq d-1$.*

A further important technique involved in the proofs is monodromy. In particular, we consider the generically finite dominant map $\pi_1 \colon \Gamma \longrightarrow Y$ projecting a correspondence Γ on the first factor, and we study the action of the monodromy group of π_1 on the generic fiber. Finally, an important role is played by Abel's theorem and some basic facts of Brill-Noether theory.

To conclude this section, we would like to note that the results on the degree of gonality we are going to present descend from the use of very same techniques.

3 Deformation and gonality of curves lying on second symmetric products of curves

Let us consider the second symmetric product $C^{(2)}$ of a smooth complex projective curve C of genus $g \geq 0$. The first issue we want to discuss is the existence of rational and hyperelliptic curves lying on $C^{(2)}$, under the assumption on C of being very general in its moduli space $\mathcal{M}_g$. Here elliptic curves are consider as special cases of hyperelliptic curves.

Clearly, if the genus of C is $g \leq 2$, then the second symmetric products contains both rational and hyperelliptic curves. When $g = 0$ we have that $C^{(2)} \cong \mathbb{P}^2$ and the claim easily follows. If $g = 1$, the surface $C^{(2)}$ is birational to $C \times \mathbb{P}^1$ by Abel's theorem and hence $C^{(2)}$ is covered by rational curves and by copies of C. Finally, if C is a genus two curve, its second symmetric product contains a unique rational curve - that is the pullback of the g^1_2 via the Abel's map - and it is covered by copies of C as well.

Then let us assume that C is a very general curve of genus $g \geq 3$. Under this assumption, we have that C is non-hyperelliptic. Hence its second symmetric product $C^{(2)}$ does not contain rational curves and it embeds into the Jacobian variety $J(C)$ via the Abel map. In particular, we can focus on curves lying on the Jacobian variety $J(C)$.

In [12], Pirola studies rigidity and existence of curves of small geometric genus on generic Kummer varieties. As a consequence of the main theorem he deduces that a generic Abelian variety of dimension $q \geq 3$ does not contain hyperelliptic curve of any genus. Since any three-dimensional Abelian variety admits an isogeny to a Jacobian variety of a genus three curve, we deduce that for any very general curve C of genus three, its Jacobian variety $J(C)$ does not contain hyperelliptic curves. Thus by using a degeneration argument we have the following (see [40, Proposition 4]).

Proposition 3.1 *If C is a very general curve of genus $g \geq 3$, the Jacobian variety $J(C)$ does not contain hyperelliptic curves.*

As a consequence of the proposition, the following holds (see [40, Lemma 5]).

Corollary 3.2 *Let C be a very general curve of genus $g \geq 3$. Then there are neither rational curves nor hyperelliptic curves lying on $C^{(2)}$.*

Now, let us turn to the problem of computing the degree of gonality of the second symmetric product $C^{(2)}$ of a smooth complex projective curve C. Thanks to the discussion above, it is easy to check that $d_o(C^{(2)}) = 1$ when the curve is either rational or elliptic, and $d_o(C^{(2)}) = 2$ for any curve of genus two. In the cases of higher genera we prove the following.

Theorem 3.3 *Let C be a smooth complex projective curve of genus $g \geq 4$. For a positive integer d, let $\mathcal{E} = \{E_t\}_{t \in T}$ be a family of curves on $C^{(2)}$ parametrized over a smooth variety T, such that the generic fiber E_t is an irreducible d-gonal curve and for any point $P \in C^{(2)}$ there exists $t \in T$ such that $P \in E_t$. Then*

$$d \geq gon(C).$$

Moreover, under the further assumption $g \geq 6$ and $Aut(C) = \{Id_C\}$, we have that equality holds if and only if E_t is isomorphic to C

Furthermore, for any smooth complex projective curve C, its second symmetric product is covered by the family of curves $\mathcal{X} = \{X_p\}_{p \in C}$ parametrized over C, where $X_p := \{p + q \in C^{(2)} \mid q \in C\}$ is isomorphic to C. Thus we deduce the following.

Theorem 3.4 *Let C be a smooth complex projective curve of genus $g \geq 4$. Then $d_o(C^{(2)}) = gon(C)$.*

We would like to note that the theorem above resolves the problem of computing the degree of gonality, except for $g = 3$. In this case we conjecture that an analogous assertion should hold.

Now, let us assume that C is a very general curve in the moduli space $\mathcal{M}_g$. As an application of Theorem 3.3 above, we dealt with the description of the cone $Nef\,(C^{(2)})_{\mathbb{R}}$ of all numerically effective $\mathbb{R}$-divisors classes in the Néron-Severi space $N^1(C^{(2)})_{\mathbb{R}}$. Since $Nef\,(C^{(2)})_{\mathbb{R}}$ is a two-dimensional convex cone, such a problem is reduced to determine the two boundary rays. The first one is the dual ray of the diagonal divisor class via the intersection pairing, which is spanned by the numerical equivalence class $(g-1)x - \frac{\delta}{2}$. The second boundary ray is spanned by the class $(\tau(C)+1)x - \frac{\delta}{2}$ depending on the real number

$$\tau(C) := inf \left\{ t > 0 \,\middle|\, (t+1)x - \frac{\delta}{2} \text{ is ample} \right\}.$$

Hence the problem of describing the cone $Nef\,(C^{(2)})_{\mathbb{R}}$ is equivalent to compute $\tau(C)$. Notice that if $(t+1)x - \frac{\delta}{2}$ is an ample class of $N^1(C^{(2)})_{\mathbb{R}}$, then it must have positive self intersection and hence $\tau(C) \geq \sqrt{g}$.

When the genus of the curve C is $g \leq 3$, the problem is totally understood (for details see e.g. [17, 18]). On the other hand, the cases of higher genera are governed by an important conjecture - due to Kouvidakis - asserting that the nef cone is as large as possible, that is $\tau(C) = \sqrt{g}$ for any $g \geq 4$. In [17], the conjecture has been proved when g is a perfect square. In the same paper, Kouvidakis proved also that $\tau(C) \leq \frac{g}{[\sqrt{g}]}$ for any $g \geq 5$. In [54], Ross uses a degeneration argument to prove a result connecting the real number $\tau(C)$ for a generic curve C of genus g, with Seshadri constants on second symmetric products of curves of genus $g-1$. Then he applies the latter result to improve the bound on $\tau(C)$ when C is a very general curve of genus five, i.e. $\tau(C) \leq \frac{16}{7}$. The following result provides a slight improvement of the bounds above when $5 \leq g \leq 8$.

Theorem 3.5 *Consider the rational numbers*

$$\tau_5 = \frac{9}{4}, \quad \tau_6 = \frac{32}{13}, \quad \tau_7 = \frac{77}{29} \quad \textit{and} \quad \tau_8 = \frac{17}{6}.$$

Let C be a smooth complex projective curve of genus $5 \leq g \leq 8$ and assume that C has very general moduli. Then

$$\tau(C) \leq \tau_g.$$

The argument of the proof is based on the main theorem in [54] together with the techniques used by Ross, due to Ein and Lazarsfeld (see [13]).

Moreover, to be able to provide new bounds on the real number $\tau(C)$, we have to discuss the self-intersection of moving curves on surfaces. Let us consider a smooth complex projective surface X and let $\{(E_t, x_t)\}_{t \in T}$ be a non trivial family of pointed curves covering X such that $mult_{x_t} E_t \geq m$ for any $t \in T$ and for some $m \geq 1$. In [13] the authors prove that the self-intersection of the general member the family is $E_t^2 \geq m(m-1)$. Under the additional hypothesis $m \geq 2$, Xu proved that $E_t^2 \geq m(m-1)+1$ (cf. [19]). We improve the latter bound, and our result turns out to be sharp. Namely, we have the following (see [40, Lemma 3]). We would like to note the the same result has been independently obtained by Knutsen, Syzdek and Szemberg in a recent paper (cf. [20]).

Theorem 3.6 *Let X be a smooth complex projective surface. Let T be a smooth variety and consider a family $\{(E_t, x_t)\}_{t\in T}$ consisting of a curve $E_t \subset X$ through a very general point $x_t \in X$ such that $mult_{x_t} E_t \geq m$ for any $t \in T$ and for some $m \geq 2$.*
If the central fibre E_0 is a reduced irreducible curve and the family is non-trivial, then

$$E_0^2 \geq m(m-1) + gon(E_0).$$

Then Theorem 3.5 descends by combining Theorem 3.3 and Theorem 3.6.

4 Galois closures and Lagrangian surfaces

In this section we turn to Lagrangian surfaces. We would like to note that the idea of constructing the examples we are going to present comes both from the study of degree of irrationality, and from a joint work with Gian Pietro Pirola and Lidia Stoppino, that deals with Galois closures of rational coverings and Lagrangian varieties.

Let X be a smooth complex algebraic surface and consider the homomorphism

$$\psi_2 \colon \bigwedge^2 H^0(X, \Omega_X^1) \longrightarrow H^0(X, \Omega_X^2).$$

The non-triviality of the kernel of this map leads to unexpected topological consequences. The main classical result is the Castelnuovo-de Franchis Theorem asserting that if there exist two non-zero forms $\omega_1, \omega_2 \in H^0(X, \Omega_X^1)$ such that $\omega_1 \wedge \omega_2 \neq 0$ and $\omega_1 \wedge \omega_2 \in Ker\, \psi_2$, then X admits a fibration over a curve of genus $g \geq 2$. This result has been generalized by Catanese in [21]. Moreover, if ψ_2 is not injective, the fundamental group of X turns out to be a non-abelian group (see for instance [22] and [23]) and other topological consequences have been studied in [24] in terms of topological index.

In the light of Castelnuovo-de Franchis Theorem it is interesting to study when there exist non-trivial elements of $Ker\, \psi_2$ that do not induce a fibration on X. Some examples of this situation have been presented in [25], [26] and [27].

Following [24], we say that a surface X is *Lagrangian* if there exist a map $a \colon X \longrightarrow a(X) \subset A$ of degree one into an Abelian variety A of dimension 4 and a holomorphic 2-form $\omega \in H^{2,0}(A)$ of rank 4 such that $a^*(\omega) = 0$. In that paper, the authors provide a sufficient condition for Lagrangian surfaces to have non-negative topological index, and they conjecture that the assertion holds for any Lagrangian surface (see [24, Conjecture 2]). We construct a family of examples of Lagrangian surfaces having negative topological index, hence disproving the conjecture above. In particular, the differential form ω turns out to be a non-trivial element of $Ker\, \psi_2$ which does not come from a fibration on X.

In order to produce our examples, the main point is to take the Galois closure of suitable rational maps between surfaces. In [7], Tokunaga and Yoshihara prove that the degree of irrationality of an Abelian surface S containing a curve D of genus three is $d_r(S) = 3$ (in particular, it could be the case of the Jacobian variety of a genus two curve). Since D induces a polarization of type (1,2) on the Abelian surface S, we follow the study of Barth (cf. [28]) to give a detailed description of the linear pencil induced by $|D|$. By opportunely blowing up S twelve times, we have a fibration $f \colon \overline{S} \longrightarrow \mathbb{P}^1$ and a degree three covering $\gamma \colon \overline{S} \longrightarrow \mathbb{F}_3$ of the Hirzebruch surface $\mathbb{F}_3$. We note that the general fiber F of f is a smooth curve of genus 3. Moreover, the restriction $\gamma_{|F} \colon F \longrightarrow \mathbb{P}^1$ is the projection of the canonical model of F in $\mathbb{P}^2$ from a point of the curve.

We define the surface X to be the minimal desingularization of the Galois closure of the covering γ. Thanks to the detailed study we did in [1, Chapter 6], it is then possible to compute the birational invariants of X and to present a holomorphic differential form $\omega \in \bigwedge^2 H^0(X, \Omega_X^1)$ vanishing on $H^0(X, \Omega_X^2)$ which is not induced by a fibration on X. In particular, we prove the following result, where by $\mathcal{W}(1,2)$ we denote the moduli space of smooth complex Abelian surfaces with a polarization of type $(1,2)$.

Theorem 4.1 *Let S be a smooth complex Abelian surface and let $\mathcal{L}$ be a line bundle on S providing a $(1,2)$-polarization. Suppose further that the pair $(S, \mathcal{L})$ is general in $\mathcal{W}(1,2)$. Then there exists a dominant* *degree three morphism $\overline{S} \longrightarrow \mathbb{F}_3$ from a suitable blow-up $\overline{S}$ of S to the Hirzebruch surface $\mathbb{F}_3$. The minimal desingularization X of the Galois closure of the covering is a surface of general type with invariants*

$$K_X^2 = 198 \quad c_2(X) = 102 \quad \chi(\mathcal{O}_X) = 25 \quad q = 4 \quad p_g = 28 \quad \tau(X) = -2.$$

Furthermore, X is a Lagrangian surface.

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