The relative canonical algebra of a fibred surface

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The object of the thesis is the study of some slope properties of fibred surfaces. A fibred surface is a surjective morphism \( f \) between an algebraic surface \( S \) and a smooth algebraic curve \( B \). In particular we are interested in the geography of fibred surfaces, which is, roughly speaking, the study of the ranges in which their numerical invariants can vary.

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The fundamental invariants associated to the fibration are
1. the self-intersection \( K_f^2 \) of the relative canonical divisor.
2. the relative Euler characteristic \( \chi_f = \chi(O_S) - (b - 1)(g - 1) \),

where \( b \) is the genus of the base curve \( B \), \( g \) is the genus of the general fibre and \( K_f \) is the relative canonical divisor \( K_S - f^*K_B \).

By classical results of Persson, Horikawa and Xiao we know a relation (slope inequality) between these invariants:

\[
K_f^2 \geq \frac{4(g - 1)}{g} \chi_f.
\]

Our aim is to investigate the nature of the number \( K_f^2 - \frac{4(g - 1)}{g} \chi_f \) and we will show that, for certain \( g \) and under generality assumption, this is the the number (counted with weights) of certain fibres called atoms.

Definition 1 ([1]) Let \( f : S \to B \) a surface fibration satisfying a prescribed condition on its general fibre. If there exists a rational number \( \lambda \), a finite set of fibres \( F_1, \ldots, F_n \) not satisfying the general condition and well-defined nonnegative rational numbers \( \text{Ind}(F_i) \) satisfying

\[
K_f^2 = \lambda \chi_f + \sum_{i=1}^{n} \text{Ind}(F_i)
\]

we call the relation a slope equality. The fibres with positive index \( \text{Ind} \) are called the atoms of the fibration.

Thus, the essence of a slope equality is the concentration of the global invariants of the surface on the fibres called atoms.

The techniques we use are based on the analysis of the relative canonical algebra \( \mathcal{R}(f) \) of the fibration \( f \)

\[
\mathcal{R}(f) = \bigoplus_{n \geq 0} \mathcal{R}_n
\]
where
\[ R_n = f_\ast \omega_f^\otimes n. \] (2)

Similar ideas have been applied successfully to the study of surface fibrations of genus 2 and 3. For genus 2 fibrations Horikawa ([4]) established the following slope equality:

**Theorem 2** (Horikawa) Let \( f : S \to B \) a surface fibration with fibres of genus 2. Then

\[ K_f^2 = 2\chi_f + \sum_{P \in B} H(P) \] (3)

where the Horikawa number of a genus 2 fibre germ over \( P \)

\[ H(P) = H(P) = \text{length}(\text{coker}(\text{Sym}^2 R_1 \to R_2)) \]

can be interpreted (roughly speaking) as the virtual number of 2-disconnected fibres of type \( E_1 + E_2 \) (with \( E_1, E_2 \) elliptic curves meeting transversally in one point).

Later on, in her PhD thesis Mendes Lopes completed the local analysis of the canonical algebra of curves of genus 2 and 3 (see [5]). Based on this, a work of Reid ([7]) led to a proof of the following equality:

**Theorem 3** (Reid) Let \( f : S \to B \) a surface fibration with fibres of genus 3 and suppose that the general fibre is nonhyperelliptic. Then

\[ K_f^2 = 3\chi_f + \sum_{P \in B} H(P) \] (4)

where the Horikawa number \( H(P) \) is defined as

\[ H(P) = \text{length}(\text{coker}(\text{Sym}^2 R_1 \to R_2)). \]

The aim of this thesis is to establish similar equalities in the case of genus 5 and 7 fibrations. The previous Theorems were based on the knowledge of local structure of the relative canonical algebra for genus 2 and 3 curves. More precisely, the local version of \( R(f) \) i.e. the canonical ring of a fibre

\[ R(F) = \bigoplus_{n \geq 0} H^0(F, \omega_F^\otimes n) \]

is completely understood even for singular curves (that was precisely the work of Mendes Lopes in her thesis, see [5]).

Unfortunately, when the genus of the fibre grows, a analysis of the canonical ring of a curve becomes more and more complicated, hence we decided to work under generality assumption on the fibres. This allowed us to perform the analysis of the relative canonical algebra, while if we had to deal with every possible “pathological” fibre this would have been impossible.

The fundamental request that we make to the fibration is the existence of the canonical embedding for every fibre. The reason is that the object we are really interested in is the relative canonical algebra, hence a fibred endowed with a canonical embedding behaves, for our purposes, precisely like a smooth nonhyperelliptic fibre.

We have found that the atoms of genus 5 and 7 fibrations are, respectively, the trigonal and tetragonal curves, which are the curves of gonality one less than the maximal allowed for curves of those genus. We know that the generic smooth curves has maximal gonality. Hence the assumption that we are going to make, i.e. that the generic fibre of the fibration has maximal gonality, is not too restrictive, because it applies to most of the fibred surfaces.
At first we study the local structure of the canonical ring of a Gorenstein curve. In particular we recall the characterization of the curves admitting a canonical morphism made in [2].

We investigate the problem of the generalization of Noether’s Theorem to singular canonical curves. Recall that Noether’s Theorem guarantees that, for a smooth canonical curve \( C \), the natural maps

\[
\text{Sym}^n H^0(C, \omega_C) \rightarrow H^0(C, \omega_C \otimes^n)
\]

are surjective for all \( n \geq 0 \).

The problem on whether this is true for any canonical curve is still open, but we proved it for the class of reduced curves:

**Theorem 4** Let \( C \subset \mathbb{P}^{g-1} \) be a projective, reduced, possibly singular, curve of algebraic genus \( p_a(C) = g \) with \( g \geq 3 \) such that the canonical sheaf is very ample. Then the canonical ring is generated in degree 1, or equivalently the homomorphisms

\[
\text{Sym}^n H^0(C, \omega_C) \rightarrow H^0(C, \omega_C \otimes^n)
\]

are surjective for any \( n \geq 1 \).

Thanks to this machinery, we are able to prove a slope equality for genus 5 fibrations. As mentioned before, we work under generality assumption, hence we suppose that every fibre has very ample canonical sheaf. Moreover, we ask that the generic fibre is nontrigonal and we find that the atoms in this case are precisely the trigonal curves.

**Theorem 5** Let \( S \) be a projective surface with at most Du Val singularities and \( B \) a projective smooth curve. Let \( f : S \rightarrow B \) be a surface fibration of genus 5.

Let us suppose that every fibre is 3-connected and nonhyperelliptic. Let us also suppose that the general fibre is nontrigonal.

Then

\[
K^2_f = 4\chi_f + \frac{1}{2} \text{length}(\mathcal{F})
\]

where \( \mathcal{F} \) is a skyscraper sheaf supported on the points having trigonal fibre.

Moreover \( \text{length}(\mathcal{F}_b) \) is even and strictly positive for any \( b \in B \) supporting a trigonal fibre.

Since the fibred surface we are dealing with is a relative canonical model, the relative canonical algebra encodes all its geometry. The key remark is that Noether’s Theorem and Petri’s Theorem remain valid for any genus 5 canonical curve, hence the relative canonical algebra is generated in degree \( \geq 3 \), and locally in degree \( \geq 2 \) away from the trigonal fibres. This means that the knowledge of the kernel of the maps \( \text{Sym}^n f_* \omega_f \rightarrow f_* \omega_f \otimes^n \) for \( n \geq 3 \) should tell us everything we need about the surface fibration. The central part of the proof is then bases on the analysis of such kernels.

In Theorem 5 the assumption that every fibre is 3-connected is not completely satisfying, since most of the stable singular curves are not even 2–connected. At least in the case of stable non isotrivial fibrations, we prove a generalization of Theorem 5. The study of the genus 5 stable fibrations is possible by means of the description of the trigonal locus in the Picard group of the moduli space of stable curves, and this was already done in [3]. In this case, the atoms are no more just the trigonal fibres, but also the singular fibres in \( \Delta_i \), with \( i \geq 1 \).

We finally discuss the problem of genus 7 fibrations. We work again under generality assumption and in this case the atoms are the tetragonal curves (i.e. smooth curves with a \( g^1_4 \) or limit of such curves). The main results is the following:
Theorem 6 Let $S$ be a projective surface with at most Du Val singularities and $B$ a projective smooth curve. Let $f : S \to B$ be a surface fibration of genus 7.

Let us suppose that any fibre is reduced, 3-connected and nontrigonal. Let us also suppose also that the generic fibre has gonality 5.

Then

$$K_f^2 = \frac{9}{2} \chi_f + \frac{1}{2} \text{length } \mathcal{F}$$

where $\mathcal{F}$ is a skyscraper sheaf supported on the points supporting a tetragonal fibre. In particular, over a general tetragonal fibre $F_P$, $\text{length } F_P = 4$.

The analysis is based again on the study of the relative canonical algebra. The key fact is a result by Mukai ([6]): for a canonical genus 7 curve $C \subset \mathbb{P}^6$ he studies the kernel of the map

$$\text{Sym}^2 H^0(C, \omega_C, I_C(2)) \to H^0(C, S^2 \frac{I_C}{I_C^2}(4))$$

and he characterizes the non-tetragonal curves as the ones having one dimensional, non degenerate kernel. Our proof is based on the construction of the right globalization of this map and on the computation of the correction terms which come from the degeneracy of the kernel.

As in the genus 5 case, we are able to generalize the result to the case of stable, non isotrivial fibrations. This too is possible thanks to the description of the tetragonal locus in the Picard group of $\overline{M}_7$ made in [3]. We find new kind of atoms, in particular the curves in $\Delta_i$ with $i \geq 1$.

References